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On the Schrödinger equation for the interaction $x^2 + \lambda x^2/(1 + gx^2)$

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Abstract. By forming the [6, 6] Padé approximants to the energy perturbation series for the interaction $x^2 + \lambda x^2/(1 + gx^2)$ as obtained from the hypervirial relations, the ground and the first three excited state energy levels are calculated. For $g \leq 2$, the results of the present calculation are in good agreement with those of the variational calculations. We also present another set of exact solutions to the Schrödinger equation for the interaction $x^2 + \lambda x^2/(1 + gx^2)$.

1. Introduction

The Schrödinger equation

$$[d^2/dx^2 + E - V(x)]\psi(x) = 0, \quad (1)$$

with an interaction of the type

$$V(x) = x^2 + \lambda x^2/(1 + gx^2), \quad (2)$$

has recently been studied by many authors (Mitra 1978, Kaushal 1979, Galicia and Killingbeck 1979, Bessis and Bessis 1980, Flessas 1981, Hautot 1981). As summarised by Mitra (1978), this type of potential is related to certain specific models in laser theory (Haken 1970), and also to a zero-dimension field theory with a nonlinear Lagrangian (Riskin and Vollmer 1967).

A set of exact solutions to the Schrödinger equation (1) has recently been found by Flessas (1981) under the conditions $\lambda < 0$ and $\lambda = \lambda(g)$. For more general cases, Mitra (1978) has calculated the ground and the first two excited state energy eigenvalues for the potential (2) by using the Ritz variational method in combination with the Givens–Householder algorithm. More recently, Bessis and Bessis (1980) have studied the same problem by taking advantage of a two-parameter λ - and g -scale transformation, and Hautot (1981) has used a Hill determinant method for the potential. The energy eigenvalues obtained by these variational calculations seem to converge towards the exact eigenvalues of the Schrödinger equation. Also, Galicia and Killingbeck (1979) have obtained exact values for the first three even parity states by using a finite-difference method.

On the other hand, Kaushal (1979) has obtained the asymptotic expansions for the eigenenergies and eigenfunctions of the wave equation for the potential (2) by expanding the factor $1/(1 + gx^2)$ in a power series for gx^2 . The energy eigenvalues

to the third order of g calculated by Kaushal (1979) with the Dingle scheme of correction are comparable to those of the variational calculations only for $g \ll 1$. Using perturbation theory, Bessis and Bessis (1980) have also calculated the energy eigenvalues by introducing a g -dependent scaling.

In this paper, we would like to point out that there exists another set of exact solutions to the Schrödinger equation with the potential (2), in addition to the set obtained by Flessas (1981). We would also like to show that the perturbation calculation for the interaction (2) can be done more consistently by employing the hypervirial relations (Hirschfelder 1960, Swenson and Danforth 1972) and the Padé approximant method (Baker 1965, Loeffel *et al* 1969, Killingbeck 1978, Lai 1981). In § 2 we present another set of exact solutions to the wave equation with the potential (2). In § 3 we apply the Hellman–Feynman theorem and the hypervirial theorems to equation (1), and calculate the energy perturbation series in powers of the parameter g . In § 4 we evaluate the [6,6] Padé approximants to the energy series for the ground and the first three excited states, and compare them with those of the variational and numerical calculations. Finally, we present the conclusion in § 5.

2. Exact solutions to the Schrödinger equation

Let us study the Schrödinger equation with the interaction $x^2 + \lambda x^2/(1 + gx^2)$,

$$\left(\frac{d^2}{dx^2} + E - x^2 - \frac{\lambda x^2}{1 + gx^2} \right) \psi = 0, \quad (3)$$

with $g > 0$. By supposing that the wavefunction ψ is given by

$$\psi = A \exp(-x^2/2)(1 + gx^2)(1 + a_1x^2 + a_2x^4 + \dots), \quad (4)$$

Flessas (1981) has recently found a set of exact solutions to equation (3). Substituting equation (4) into equation (3), he finds for the first two exact solutions of equation (3)

$$E_0 = 1 - g, \quad \lambda_0 = -4g - 2g^2, \quad \psi_0 = A_0 \exp(-x^2/2)(1 + gx^2), \quad (5)$$

and

$$E_2 = 3 - 7g \pm (25g^2 - 12g + 4)^{1/2}, \quad \lambda_2 = -(6g + 7g^2) \pm g(25g^2 - 12g + 4)^{1/2},$$

$$\psi_2 = A_2 \exp(-x^2/2)(1 + gx^2) \left(1 - \frac{4g^2}{12g^2 + 4g + \lambda_2} x^2 \right), \quad (6)$$

where A_0 and A_2 are normalisation constants.

We notice that, in addition to the set of equation (4), there exists another set of exact solutions to equation (3). Let us suppose that the wavefunctions of equation (3) can also be written in the form

$$\psi = A \exp(-x^2/2)(1 + gx^2)x(1 + b_1x^2 + b_2x^4 + \dots), \quad (7)$$

which are odd functions of x in contrast to equation (4). Inserting equation (7) into equation (3), we obtain, for example, for the first two solutions of equation (7)

$$E_1 = 3(1 - 2g), \quad \lambda_1 = -4g - 6g^2, \quad \psi_1 = A_1 \exp(-x^2/2)(1 + gx^2)x, \quad (8)$$

and

$$E_3 = 5 - 13g \pm (49g^2 - 4g + 4)^{1/2}, \quad \lambda_3 = -(6g + 13g^2) \pm g(49g^2 - 4g + 4)^{1/2},$$

$$\psi_3 = A_3 \exp(-x^2/2)(1 + gx^2)x \left(1 - \frac{4g^2}{20g^2 + 4g + \lambda_3} x^2 \right), \tag{9}$$

where A_1 and A_3 are normalisation constants. Here, we note that, when $\lambda = g = 0$, equations (5), (6) and (8), (9) reduce to the first four states of the harmonic oscillator.

In table 1, we present some calculated values of the energies E_i and the constants λ_i as a function of g for reference. From table 1, we see that, for the set of exact solutions to equation (3), the values of λ_i are negative, and the values of E_i satisfy

$$E_1 \leq 1, \quad E_2 \leq 3, \quad E_3 \leq 5, \quad E_4 \leq 7. \tag{10}$$

The solutions (4) and (7) thus form a set of exact solutions to the Schrödinger equation (3); this set has a one-to-one pairing with the complete set of unperturbed states for the oscillator potential (2) as $g \rightarrow 0$.

Table 1. Calculated values of the energy E and the parameter λ as a function of g for the first four states. The numbers given in the brackets correspond to the values of λ .

g	E_0 (λ_0)	E_1 (λ_1)	E_2 (λ_2)	E_3 (λ_3)
0	1 (0)	3 (0)	5 (0)	7 (0)
0.05	0.9 (-0.205)	2.7 (-0.215)	4.510 78 (-0.224 46)	6.330 53 (-0.233 47)
0.10	0.8 (-0.42)	2.4 (-0.46)	4.046 43 (-0.495 36)	5.722 38 (-0.527 76)
0.15	0.7 (-0.645)	2.1 (-0.735)	3.612 08 (-0.808 19)	5.171 91 (-0.874 21)
0.20	0.6 (-0.88)	1.8 (-1.04)	3.212 45 (-1.157 51)	4.671 56 (-1.265 69)
0.25	0.5 (-1.125)	1.5 (-1.375)	2.850 78 (-1.537 30)	4.212 22 (-1.696 95)
0.30	0.4 (-1.38)	1.2 (-1.74)	2.527 88 (-1.941 64)	3.785 15 (-2.164 46)
0.35	0.3 (-1.645)	0.9 (-2.135)	2.241 89 (-2.365 34)	3.383 00 (-2.666 95)
0.40	0.2 (-1.92)	0.6 (-2.56)	1.988 85 (-2.804 46)	3 (-3.2)
0.45	0.1 (-2.205)	0.3 (-3.015)	1.763 77 (-3.256 31)	2.631 72 (-3.765 72)
0.50	0 (-2.5)	0 (-3.5)	1.561 55 (-3.719 22)	2.274 92 (-4.362 54)

3. Hypervirial relations for the interaction $x^2 + \lambda x^2/(1 + gx^2)$

In this section, we apply the hypervirial theorems (Hirschfelder 1960) and the Hellman–Feynman theorem to the Schrödinger equation (3). The Hamiltonian H for the interaction (2) can be written as

$$H = -d^2/dx^2 + V(x), \tag{11}$$

where the potential $V(x)$ is given by

$$V(x) = x^2 + \lambda x^2 / (1 + gx^2). \quad (12)$$

Starting from the basic commutation relations, we find the following commutation relations (Hirschfelder 1960, Swenson and Danforth 1972):

$$[x^N d/dx, H] = x^N dV/dx + 2Nx^{N-1}(V-H) + \frac{1}{2}N[x^{N-1}, H] - \frac{1}{2}N(N-1)(N-2)x^{N-3}, \quad N \geq 0. \quad (13)$$

The hypervirial theorems then require that $\langle [x^N, H] \rangle$ and $\langle [x^N d/dx, H] \rangle$ for $N \geq 0$ vanish for the eigenstates of equation (3). We obtain, from equation (13), the hypervirial relations (Swenson and Danforth 1972)

$$E \langle x^N \rangle = \frac{1}{2}(N+1)^{-1} \langle x^{N+1} dV/dx \rangle + \langle x^N V \rangle - \frac{1}{4}N(N-1) \langle x^{N-2} \rangle. \quad (14)$$

The potential (12), in accordance with the perturbation scheme (Kaushal 1979), can be expressed as

$$V(x) = \sum_{k=0}^{\infty} g^k V_k x^{2(k+1)}, \quad (15)$$

where the potential coefficients V_k are

$$V_0 = \omega^2 = 1 + \lambda, \quad V_j = \lambda(-1)^j, \quad j \neq 0. \quad (16)$$

We note here that the series (15) is valid only for $gx^2 < 1$, and the energy series E obtained from equation (14) will be at best asymptotic. However, one may use the hypervirial Padé method (Killingbeck 1978, Lai 1981) to sum such asymptotic series. Let us assume that the energy E and the expectation values $\langle x^j \rangle$ take the forms (Killingbeck 1978, Grant and Lai 1979)

$$E_n = \sum_{k=0}^{\infty} E_n^{(k)} g^k, \quad (17)$$

$$\langle x^j \rangle = \sum_{k=0}^{\infty} C_j^{(k)} g^k \quad (18)$$

where g is the perturbation parameter. The unperturbed value of $E_n^{(0)}$ in equation (17) is given by

$$E_n^{(0)} = \omega q, \quad q = 2n + 1, \quad (19)$$

where n is the quantum number for the harmonic oscillator. The unperturbed values of $C_j^{(0)}$ in equation (18), on the other hand, can be obtained from the hypervirial equations (14). From the Hellman-Feynman theorem, we find then (Grant and Lai 1979)

$$kE_n^{(k)} = \sum_{m=1}^k m V_m C_{2m+2}^{(k-m)}. \quad (20)$$

By inserting equations (15)–(18) into equation (14), we can determine the energy coefficients $E_n^{(k)}$ in a hierarchical manner with the aid of equations (19)–(20) (Grant and Lai 1979). For example, we obtain

$$E_n^{(1)} = V_1 C_4^{(0)} = -(3\lambda/8\omega^2)(q^2 + 1), \quad (21)$$

$$E_n^{(2)} = \frac{1}{2}(V_1 C_4^{(1)} + 2V_2 C_6^{(0)}) = -\frac{\lambda^2 q}{64\omega^5}(17q^2 + 67) + \frac{5\lambda q}{16\omega^3}(q^2 + 5), \quad (22)$$

$$\begin{aligned} E_n^{(3)} &= \frac{1}{3}(V_1 C_4^{(2)} + 2V_2 C_6^{(1)} + 3V_3 C_8^{(0)}) \\ &= -\frac{3\lambda^3}{1024\omega^8}(125q^4 + 1138q^2 + 513) + \frac{225\lambda^2}{256\omega^6}(11q^4 + 118q^2 + 63) \\ &\quad - \frac{35\lambda}{128\omega^4}(q^4 + 14q^2 + 9), \end{aligned} \quad (23)$$

$$\begin{aligned} E_n^{(4)} &= \frac{1}{4}(V_1 C_4^{(3)} + 2V_2 C_6^{(2)} + 3V_3 C_8^{(1)} + 4V_4 C_{10}^{(0)}) \\ &= -\frac{\lambda^4 q}{16384\omega^{11}}(10689q^4 + 178330q^2 + 305141) \\ &\quad + \frac{\lambda^3 q}{2048\omega^9}(3129q^4 + 59110q^2 + 11728) \\ &\quad - \frac{\lambda^2 q}{1024\omega^7}(1149q^4 + 25930q^2 + 61361) + \frac{63\lambda q}{256\omega^5}(q^4 + 30q^2 + 89), \end{aligned} \quad (24)$$

where $\omega^2 = 1 + \lambda$ and $q = 2n + 1$. Expressing in terms of powers of $h = q/2(1 + \lambda)^{1/2}$ (Kaushal 1979), the energy eigenvalue E_n is then given by

$$\begin{aligned} E_n &= q(1 + \lambda)^{1/2} - \frac{3(q^2 + 1)}{4(1 + \lambda)^{1/2}} h \\ &\quad + \frac{\lambda q}{16(1 + \lambda)^{1/2}} \left(-\frac{\lambda}{1 + \lambda}(17q^2 + 67) + 20(q^2 + 5) \right) h^2 \\ &\quad + \frac{\lambda}{128(1 + \lambda)^{1/2}} \left(-\frac{3\lambda^2}{(1 + \lambda)^2}(125q^4 + 1138q^2 + 513) \right. \\ &\quad \left. + \frac{900\lambda}{1 + \lambda}(11q^4 + 118q^2 + 63) - 280(q^4 + 14q^2 + 9) \right) h^3 \\ &\quad + \frac{q\lambda}{1024(1 + \lambda)^{1/2}} \left(-\frac{\lambda^3}{(1 + \lambda)^3}(10689q^4 + 178330q^2 + 305141) \right. \\ &\quad \left. + \frac{8\lambda^2}{(1 + \lambda)^2}(3129q^4 + 59110q^2 + 11728) \right. \\ &\quad \left. - \frac{16\lambda}{1 + \lambda}(1149q^4 + 25930q^2 + 61361) + 4032(q^4 + 30q^2 + 89) \right) h^4 + \dots \end{aligned} \quad (25)$$

We note here that the first four terms in the expression (25) are in agreement with those obtained by Kaushal (1979). The solution (25) should be valid for $h \ll 1$.

Our hyperviral scheme for calculating the energy coefficients $E_n^{(k)}$ for the interaction $x^2 + \lambda x^2 / (1 + gx^2)$ can be more conveniently done by computer (Grant and Lai 1979). We present, for reference, some results for perturbative calculations for the ground state energy for the case $g = 1.0$, $\lambda = 1.0$. Denoting that $E(k)$ is the energy series sum up to the g^k term, we find, for $g = 1.0$ and $\lambda = 1.0$, $E(0) = 1.4142$, $E(1) = 1.0392$, $E(2) = 1.4701$, $E(3) = 0.9105$, $E(4) = 1.7045$, ..., $E(11) = -787.3805$ and

$E(12) = 4200.5040$. The result of Kaushal (1979) for the same calculation correct to the third order of g with the Dingle scheme of correction is $E(3) = 1.227$. The exact value from the calculations of Mitra (1978) and Galicia and Killingbeck (1979) is 1.2324. It is evident that the perturbative calculation of the energy E_n is not reliable when $g \sim 1$. However, the results of the perturbative calculation can be much improved with the use of the Padé approximant method (Baker 1965, Loeffel *et al* 1969, Killingbeck 1978, Lai 1981).

4. The Padé approximant calculation of energy eigenvalues

The energy series (17) for the interaction $\lambda x^2/(1+gx^2)$ appears to be asymptotic for $g \sim 1$. However, we can still calculate the energy E_n for $g \sim 1$ to a very high accuracy by forming the Padé approximants to the energy series (Killingbeck 1978, Lai 1981). The $[N, N]$ Padé approximant to the energy series is given by (Baker 1965)

$$E[N, N] = E^{(0)} \frac{1 + g p_1 + \dots + g^N p_N}{1 + g q_1 + \dots + g^N q_N} = E^{(0)} + g E^{(1)} + g^2 E^{(2)} + \dots + g^{2N} E^{(2N)}, \quad (26)$$

where the coefficients q_1, \dots, q_N and p_1, \dots, p_N can be found from the knowledge of $E^{(1)}, E^{(2)}, \dots, E^{(2N)}$, which can be computed from the hypervirial relations in a hierarchical manner. We confine ourselves to the calculation of the Padé approximant $E[6, 6]$ in this paper.

Our calculated energy values E_n of the $[6, 6]$ Padé approximants to the energy series (17) for the ground and the first three excited states are shown in table 2. Our results are compared with those of Bessis and Bessis (1980) which are listed in the brackets in table 2. The first three energy levels for different values of g and λ were also calculated by Mitra (1978) by using the Ritz variational method with the Jacobi diagonalisation of more than 30×30 matrices. We note here that our calculated values of E_n in table 2 are closer to the results of Mitra than those obtained by Bessis and Bessis from the Jacobi diagonalisation of 18×18 matrices. To show this point, we present, in table 3, the results of energy eigenvalues for $g = 1, 2$ and $\lambda = 0.1, 0.5, 1.0$ as calculated by Mitra (1978), Bessis and Bessis (1980), Galicia and Killingbeck (1979) and the present method for comparison. It is evident from table 3 that our results are in better agreement with those of Mitra and of Galicia and Killingbeck than with those of Bessis and Bessis in this region. Our work is therefore an improvement over the perturbative scheme of Kaushal (1979).

From our calculations, we thus conclude that, for $g \leq 2$, the $[6, 6]$ Padé approximants to the energy series (17) can be used to determine the energy eigenvalues of the potential (2) to a very high accuracy. However, for $g > 2$, either the higher-order Padé approximants or the variational method must be used to calculate the various energy eigenvalues. Our scheme can also be used to evaluate energy eigenvalues for the case $\lambda < 0$ as discussed in § 2.

5. Conclusion

In this paper, we have presented another set of exact solutions to the Schrödinger equation with the potential $V(x) = x^2 + \lambda x^2/(1+gx^2)$ for $\lambda < 0$, in addition to the set of exact solutions proposed by Flessas (1981). We have also calculated the $[6, 6]$ Padé

Table 2. The first four energy levels calculated from the Padé approximants $E[6,6]$. The numbers in the brackets correspond to the variational results of Bessis and Bessis.

$g \backslash \lambda$	0.1	0.5	1.0	2.0	5.0	10.0	50.0	100.0
0.1	1.043 173 71 (1.043 173 71)	1.203 039 55 (1.203 039 55)	1.380 531 80 (1.380 531 80)	1.685 617 40 (1.685 617 40)	2.389 541 55 (2.389 541 55)	3.250 261 22 (3.250 261 22)	7.068 696 47 (7.068 696 47)	9.976 180 09 (9.976 180 09)
	3.120 081 86 (3.120 081 86)	3.570 809 29 (3.570 809 29)	4.079 883 01 (4.079 883 01)	4.968 599 33 (4.968 599 33)	7.050 963 92 (7.050 963 92)	9.619 066 41 (9.619 066 41)	21.060 738 26 (21.060 738 3)	29.781 191 11 (29.781 191 1)
	5.181 094 79 (5.181 094 79)	5.871 583 70 (5.871 583 70)	6.667 919 10 (6.667 919 10)	8.086 804 04 (8.086 804 04)	11.484 808 63 (11.484 808 63)	15.729 336 34 (15.729 336 3)	34.763 829 69 (34.763 829 69)	49.292 690 50 (49.292 690 5)
	7.231 009 98 (7.231 009 98)	8.121 871 44 (8.121 871 44)	9.166 567 47 (9.166 567 47)	11.062 748 62 (11.062 748 6)	15.706 662 07 (15.706 662 1)	21.591 005 51 (21.591 005 5)	48.181 498 19 (48.181 498 2)	68.513 062 23 (68.513 052 2)
0.5	1.031 214 32 (1.031 214 32)	1.151 563 40 (1.151 563 40)	1.292 950 42 (1.292 950 42)	1.551 049 13 (1.551 049 13)	2.192 118 47 (2.192 118 47)	3.016 854 29 (3.016 854 29)	6.792 789 63 (6.792 789 63)	9.692 157 81 (9.692 157 82)
	3.073 902 58 (3.073 902 58)	3.363 801 56 (3.363 801 56)	3.713 902 25 (3.713 902 25)	4.376 583 05 (4.376 583 05)	6.121 059 23 (6.121 059 23)	8.482 270 75 (8.482 270 75)	19.685 037 65 (19.685 037 6)	28.362 597 95 (28.362 597 9)
	5.093 069 15 (5.093 069 15)	5.463 211 42 (5.463 211 42)	5.920 632 64 (5.920 632 64)	6.815 287 11 (6.815 287 11)	9.320 760 67 (9.320 760 67)	12.948 038 91 (12.948 038 91)	31.238 042 27 (31.238 042 3)	45.636 572 89 (45.636 572 9)
	7.105 838 62 (7.105 838 62)	7.527 888 60 (7.527 888 60)	8.052 438 85 (8.052 438 85)	9.089 127 68 (9.089 127 68)	12.092 348 02 (12.092 348 02)	16.679 411 96 (16.679 411 96)	41.549 253 05 (41.549 253 05)	61.577 873 22 (61.577 873 2)
1.0	1.024 123 42 (1.024 123 42)	1.118 552 87 (1.118 552 87)	1.232 353 53 (1.232 353 53)	1.447 324 61 (1.447 324 61)	2.013 002 02 (2.013 002 02)	2.782 330 54 (2.782 330 54)	6.478 114 96 (6.478 114 96)	9.359 418 03 (9.359 418 03)
	3.051 526 02 (3.051 526 02)	3.255 802 59 (3.255 802 59)	3.507 397 06 (3.507 397 06)	3.998 466 74 (3.998 466 74)	5.379 550 33 (5.379 550 33)	7.417 672 06 (7.417 672 06)	18.128 712 18 (18.128 712 2)	26.705 965 63 (26.705 965 6)
	5.058 990 012 (5.058 990 012)	5.294 916 27 (5.294 916 27)	5.589 833 55 (5.589 833 55)	6.178 209 08 (6.178 209 08)	7.920 591 92 (7.920 591 92)	10.704 480 59 (10.704 480 59)	27.375 341 08 (27.375 341 08)	41.441 099 75 (41.441 099 75)
	7.064 966 64 (7.064 966 64)	7.324 464 57 (7.324 464 57)	7.649 068 99 (7.649 068 99)	8.283 062 37 (8.283 062 37)	10.157 017 71 (10.157 017 71)	13.390 003 25 (13.390 003 25)	34.645 158 69 (34.645 158 69)	53.839 092 96 (53.839 092 96)
2.0	1.017 281 60 (1.017 281 60)	1.085 285 78 (1.085 285 78)	1.168 723 92 (1.168 723 92)	1.330 767 28 (1.330 767 28)	1.782 167 01 (1.782 167 01)	2.442 502 65 (2.442 502 65)	5.931 989 63 (5.931 989 63)	8.758 278 63 (8.758 278 63)
	3.032 957 27 (3.032 957 27)	3.163 714 18 (3.163 714 18)	3.326 136 99 (3.326 136 99)	3.649 162 07 (3.649 162 07)	4.596 899 98 (4.596 899 98)	6.149 399 12 (6.149 399 12)	15.497 538 76 (15.497 538 76)	23.743 326 04 (23.743 326 04)
	5.034 551 93 (5.034 551 93)	5.172 579 32 (5.172 579 32)	5.345 642 68 (5.345 642 68)	5.690 305 21 (5.690 305 21)	6.703 210 68 (6.703 210 68)	8.693 472 13 (8.693 472 13)	21.387 895 59 (21.387 895 59)	34.257 741 24 (34.257 741 24)
	7.037 758 87 (7.037 758 87)	7.188 010 61 (7.188 010 61)	7.378 339 51 (7.378 339 51)	7.699 343 31 (7.699 343 31)	8.152 093 68 (8.152 093 68)	10.357 529 88 (10.357 529 88)	25.016 405 91 (25.016 405 91)	41.492 824 03 (41.492 824 03)

Table 3. Calculated values of the energy E_n obtained from four different methods: (a) Mitra, (b) Bessis and Bessis, (c) Galicia and Killingbeck† and (d) the present work.

λ		0.1	0.5	1.0
$n = 1$	a	1.024 10	1.118 54	1.232 35
	b	1.024 186 75	1.118 589 46	1.232 372 05
	c	1.024 109 61	1.118 545 99	1.232 350 72
	d	1.024 123 42	1.118 552 87	1.232 353 53
$g = 1 \ n = 2$	a	3.051 49	3.255 77	3.507 38
	b	3.051 650 67	3.255 842 10	3.507 420 53
	d	3.051 526 02	3.255 802 59	3.507 397 06
	a	5.034 44	5.294 88	5.589 77
$n = 3$	b	5.059 286 55	5.295 062 92	5.589 860 86
	c	5.058 963 28	5.294 888 95	5.589 778 94
	d	5.058 990 12	5.294 916 27	5.589 833 55
	a	1.017 18	1.085 19	1.168 67
$n = 1$	b	1.017 894 66	1.087 064 9	1.170 485
	c	1.017 180 29	1.085 196 21	1.168 671 06
	d	1.017 281 60	1.085 285 78	1.168 723 92
	a	3.032 76	3.163 46	3.326 02
$g = 2 \ n = 2$	b	3.031 773	3.186 776	3.329 042
	d	3.032 957 27	3.163 714 18	3.326 136 99
	a	5.034 44	5.172 40	5.345 24
	b	5.035 846	5.175 886	5.348 490 66
$n = 3$	c	5.034 441 87	5.172 399 81	5.345 248 72
	d	5.034 551 93	5.172 579 32	5.345 642 68

† Corrected results provided by the authors.

approximants to the energy series (17) as derived from the hypervirial relations. The results of the present calculation for the ground and the first three excited state energy levels for $g \leq 2$ are in very good agreement with those of the variational calculations by Mitra (1978) and Bessis and Bessis (1980), and with those computed by the finite-difference method of Galicia and Killingbeck (1979). In comparison with the variational method, the present hypervirial Padé scheme is simpler and more straightforward, and can be used to check the results of the variational calculations for $g \leq 2$.

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